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Effects of Orbital Parameter Uncertainties

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A number of important space flight mechanics problems require accurate uncertainty estimates of the position and velocity of an orbiting object. The lack of information might simply derive from a tracking error or from a propagation error, and in many cases it can be of a significant magnitude to the extent that many objects orbiting around our planet are on orbits from which very little can be inferred, in particular, space debris. To deal with these kinds of problems, both astrodynamicists and engineers have developed many different techniques and models, many of which date back to the 1970s. E. Öpik and N. G. Dennis laid the basis of the primary theories that are used to deal with these problems. Both the theories, though, require restrictive hypotheses that limit their use to particular cases. In this work, an original and general approach to the problem of dealing with uncertainties in some or in all of the six orbital parameters is proposed. No restrictive hypothesis is introduced. The case of osculating orbital parameters is then considered, and a set of six new formulas is derived. One of these six expressions is shown to represent a generalization of the Dennis statistical theory of satellite motion.

Nomenclature

a	=	orbit semimajor axis
b	=	orbit semiminor axis
E	=	eccentric anomaly
e	=	orbit eccentricity
f	=	probability density function of the argument of perigee
G	=	probability density function of the eccentricity
g	=	probability density function of the right ascension of the ascending node
H	=	probability density function of the semimajor axis
i	=	orbit inclination
k	=	spring constant
L	=	probability density function of the inclination
m	=	mass of the spring-mass system
p	=	orbit parameter
r, σ, λ	=	spherical coordinates
r_a	=	apogee radius
r_p	=	perigee radius
$\hat{r}(t), \hat{\sigma}(t), \hat{\lambda}(t)$	=	Kepler's problem solution
$\hat{r}(\sigma)$	=	radius along an orbit at longitude σ
$\hat{r}_i(\lambda)$	=	radii along an orbit at latitude λ
x	=	mass position in the spring-mass system
δ	=	Dirac delta
$\hat{\theta}(\sigma)$	=	angle between the position vector along a Keplerian orbit and the line of nodes as a function of the longitude
$\hat{\lambda}(\sigma)$	=	latitude along an orbit at longitude σ
$\hat{\lambda}_i(r)$	=	latitudes along an orbit at radius r
ν	=	true anomaly
$\hat{\sigma}_i(r)$	=	longitudes along an orbit at radius r
$\hat{\sigma}_i(\lambda)$	=	longitudes along an orbit at latitude λ
τ	=	time of pericenter passage
Ω	=	right ascension of the ascending node
ω	=	argument of perigee
$\tilde{\omega}$	=	pulsation of the spring-mass system

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Introduction

THE number of objects orbiting around our planet has been increasing at a roughly constant rate since the very beginning of the space era. This great number of objects whose motion is described by complex and time-consuming computations suggests a statistical approach to several orbital mechanics problems.[1−7] Dennis,[4] in his cornerstone paper, laid down the basis for such an approach. The use of the Dennis statistical approach to construct a spatial density function related to a large family of satellites [low-Earth orbit (LEO) or geostationary Earth orbit (GEO)] has later been shown by Chobotov[5] and many other authors. Despite the number of published articles, not much has changed in the theory in the past few decades. The limits of Dennis theory are located in the impossibility of considering nonuniform distribution for the argument of perigee and for the longitude of the ascending node and in its being limited to the sole case of uncertainties on these two parameters. This work seeks to introduce a general approach to these kinds of problems, an approach that allows us to generalize the commonly used methodology and to derive a new set of six equations describing the effects of uncertainties on the orbital parameters. These formulas are quite useful in connection with the problem of the space debris modeling. In particular, given any set of orbiting objects, they allow the construction of an accurate object spatial density, a function that is directly related to the probability of finding an object in a given region of space. Even Molniya satellites and geostationary satellites can be satisfactorily modeled by these equations despite the clustering of the argument of perigee and the right ascension of the ascending node around some particular values.

Simple Example

The novel methodology used is based on some properties of the Dirac delta, and in particular on the theorem that allows a change of variable in these generalized functions. Suppose we want to evaluate the following expression:

$$\int_a^b f[x(t)]\delta[x(t) - \hat{x}]\,\mathrm{d}t = ?$$

where $\hat{x}$ is an arbitrary constant. Such an integral can be evaluated using a fundamental theorem on the generalized functions[8]:

$$\delta[x(t) - \hat{x}] = \sum_n \frac{1}{|\mathrm{d}x/\mathrm{d}t|_{t=t_n^*}}\delta\left[t - t_n^*\right] \tag{1}$$

where t_n^* is the nth solution to the equation $x(t) = \hat{x}$. The result can now be found by applying the definition of the Dirac delta:

$$\int_a^b f[x(t)]\delta[x(t)-\hat{x}]\,\mathrm{d}t = \sum_j \frac{1}{|\mathrm{d}x/\mathrm{d}t|_{t=t_j^*}} f\left[x\left(t_j^*\right)\right]$$

where we must consider only those solutions $t_j^* \in [a, b]$. Consider now a continuous random variable x described by a probability density function (PDF) f and let $y(x)$ be a differentiable transformation. The problem of finding the PDF of y can be easily solved via a standard change of variable technique[9] whenever $y(x)$ is a one-to-one transformation. If this does not happen, things become more complicated, and all of the different intervals of monotonicity must be considered separately leading to very lengthy operations. This paper shows an alternative technique based on the manipulation of the Dirac deltas that simplifies the calculations. The technique, independently developed by the author, is also the main result of a previous work by Au and Tam,[10] where the advantages of this method with respect to the traditional one are also discussed.

We describe this technique with a very simple case. Consider a mass-spring system defined by a mass m connected to a linear elastic spring of elastic constant k and the pdf related to the question, "Where is the mass?" Such a PDF depends on the information we have on the phenomenon. In the case of the mass-spring system, the information are the time instant t, the amplitude A of the motion, and the phase φ (or an alternative set of two parameters). Once these values are known, we will be able to write the mass position through the expression $\hat{x}(t) = A\cos(\tilde{\omega}t + \varphi)$, where $\tilde{\omega} = \sqrt{(k/m)}$. Because the deterministic distribution function of the variable x is clearly a step function and because the derivative of the step function is a Dirac delta, we can write the following expression for the PDF:

$$\rho(x|t, A, \varphi) = \delta[x - \hat{x}(t, A, \varphi)]$$

Note that in this deterministic situation the probability of the mass being in a certain interval I can be evaluated as 1 or 0 according to the mass being in I or not.

If we do not know how long ago the motion has started, we must consider the time to be a random variable and carry out what is usually called a randomization of the process:

$$\rho(x|A, \varphi) = \frac{1}{T}\int_0^T \delta[x - \hat{x}(t, A, \varphi)]\,\mathrm{d}t$$

[The term "randomization" is actually used only by part of the scientific community to indicate the application of the identity $p(x) = \int p(x|t)p(t)\,\mathrm{d}t$. Others prefer to use the term "disintegration," a term that somehow underlies how the conditioning events are "taken away" and therefore disappear in the final probability. Another common way of referring to this standard procedure is to say that the theorem of total probabilities is applied. This last nomenclature generates some degree of confusion as the expression

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

with E_i forming a noncompatible set of events, is called theorem of the total probabilities.] In the preceding equation we have considered all of the time instants to be equally probable within the interval $[0, T]$, where T represents the period of the harmonic motion. We can now evaluate this integral with the aid of the theorem expressed by Eq. (1):

$$\rho(x|A, \varphi) = \frac{\tilde{\omega}}{2\pi}\sum_{i=1}^{2} \frac{1}{A\tilde{\omega}|\sin(\tilde{\omega}t + \varphi)|_{t=t_i^*}}$$

where $t_i^* \in [0, T]$ are the two solutions in t of the equation $x = A\cos(\tilde{\omega}t + \varphi)$. Hence we get

$$\rho(x|A, \varphi) = \frac{1}{2\pi}\sum_{i=1}^{2} \frac{1}{A\sqrt{1 - \cos^2(\tilde{\omega}t + \varphi)}}\Bigg|_{t=t_i^*} = \frac{1}{\pi}\frac{1}{\sqrt{A^2 - x^2}} = \rho(x|A) \tag{2}$$

Notice that the phase φ is no longer conditioning the possible position of the mass and has therefore been removed from the conditioning variables. With the aid of this methodology, we can actually go further in considering a generic PDF associated with the randomized variable t. If we denote such a PDF with the letter f, the preceding expression becomes

$$\rho(x|A, \varphi) = \frac{1}{\tilde{\omega}}\frac{1}{\sqrt{A^2 - x^2}}\sum_{i=1}^{2} f\left(t_i^*\right) \tag{3}$$

When $f = \tilde{\omega}/2\pi$, Eq. (3) turns into Eq. (2). We observe that in this case the phase does appear in the expression through the solutions t_i^*. Equations (2) and (3) do not give any kind of information on the velocity of the mass as we did not consider it in the starting expression. It is not difficult to include such a variable by writing

$$\rho(x, v|t, A, \varphi) = \delta_x[x - \hat{x}(t, A, \varphi)]\delta_v[v - \hat{v}(t, A, \varphi)] \tag{4}$$

where $\hat{x}(t) = A\cos(\tilde{\omega}t + \varphi)$ and $\hat{v}(t) = -A\tilde{\omega}\sin(\tilde{\omega}t + \varphi)$. This expression contains the product of two Dirac deltas. A discussion of such a mathematical operation can be found in Raju.[11] In this paper I will not discuss the details of a rigorous mathematical treatment of the generalized functions as it would obscure the real aim of this work that is to be found in the outline of a simple procedure to deal with probabilistic problems related to Keplerian mechanics. The reader can refer to the literature that tries to justify the mathematical use of these symbols often used in physical applications.

Consider now Eq. (4). If the time is randomized, assuming a constant PDF, we can write

$$\rho(x, v|A, \varphi) = \frac{1}{T}\int_o^T \delta_x[x - \hat{x}(t, A, \varphi)]\delta_v[v - \hat{v}(t, A, \varphi)]\,\mathrm{d}t$$

Evaluating analytically and exploiting the theorem on the change of variable in a Dirac delta, we find

$$\rho(x, v|A) = (1/2\pi)\left(1\big/\sqrt{A^2 - x^2}\right)(\delta_v[v - \hat{v}_1] + \delta_v[v - \hat{v}_2])$$

where $\hat{v}_{1,2} = \pm\tilde{\omega}\sqrt{(A^2 - x^2)}$. We can now consider randomizing the amplitude A. If the PDF associated is denoted by the letter g and changing again variable in the Dirac delta,

$$\rho_A(x, v) = (1/\pi)\left(1\big/\sqrt{v^2 + \tilde{\omega}^2 x}\right)g\left(\sqrt{x^2 + v^2/\tilde{\omega}^2}\right) \tag{5}$$

Another interesting expression can be written if we consider some degree of uncertainty on $\tilde{\omega}$ rather than on the amplitude, in which case we get

$$\rho_{\tilde{\omega}}(x, v) = (1/\pi)[1/(A^2 - x^2)]h\left(v\big/\sqrt{A^2 - x^2}\right) \tag{6}$$

In Eq. (6), the PDF associated with $\tilde{\omega}$ has been indicated with the letter h.

Equations (5) and (6) have been obtained applying Eq. (1) to the deterministic PDF written in terms of Dirac deltas. The same approach can be used to study the statistical properties of any dynamical systems of dimension N, in which case we will have N Dirac deltas that can be eliminated by randomizing an equal number of parameters. (These could be the initial conditions, the time or any other set of parameters uniquely determining the evolution of the system.)

Note that the same expressions derived here can be obtained by using a classic change of variable technique.[9] In particular, set $x = A\cos(\tilde{\omega}t + \phi)$, $v = -A\tilde{\omega}\sin(\tilde{\omega}t + \phi)$, and consider this to be a transformation from A, ϕ to x, v. This transformation is not monotonic, and some care must be taken in evaluating the final PDF. We observe that if the transformation is treated as monotonic the final expression result differs from Eq. (5) by a factor of 2. As the complexity of the case increases, so do the benefits of using Dirac deltas in the algebraic manipulations.[10]

Keplerian Case with Osculating Elements

The mathematical derivations shown for the mass-spring dynamical system can be done for the more complex case of the Kepler's problem (and therefore of the two-body problem). In this case the osculating parameters will uniquely determine a solution to the problem and will therefore be considered as the conditioning variables when writing the PDF for the deterministic case:

$$\rho(r,\sigma,\lambda|t,e,a,\tau,\omega,\Omega,i) = \delta_r[r-\hat r(t)]\delta_\sigma[\sigma-\hat\sigma(t)]\delta_\lambda\lfloor\lambda-\hat\lambda(t)\rfloor \tag{7}$$

Note that spherical coordinates have been used. In the preceding expression $\hat r(t)$, $\hat\sigma(t)$, $\hat\lambda(t)$ represent the solution of Kepler's problem with initial conditions determined by the orbital parameters conditioning the event. We can now start considering the randomization of the time t:

$$\rho(r,\sigma,\lambda|e,a,\omega,\Omega,i) = \frac{1}{T}\int_0^T \delta_r[r-\hat r(t)]\delta_\sigma[\sigma-\hat\sigma(t)]\delta_\lambda[\lambda-\hat\lambda(t)]\,\mathrm{d}t$$

where for the Keplerian case $T = 2\pi\sqrt{(a^3/\mu)}$. Three equivalent expressions can be obtained depending on the Dirac delta we decide to eliminate through the use of Eq. (1). To do so, it will be necessary to evaluate the absolute values of the following derivatives:

$$\left.\frac{\mathrm{d}\hat r(t)}{\mathrm{d}t}\right|_{\hat r=r}, \qquad \left.\frac{\mathrm{d}\hat\sigma(t)}{\mathrm{d}t}\right|_{\hat\sigma=\sigma}, \qquad \left.\frac{\mathrm{d}\hat\lambda(t)}{\mathrm{d}t}\right|_{\hat\lambda=\lambda}$$

We therefore write

$$\left|\frac{\mathrm{d}\hat r(t)}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}\hat r}{\mathrm{d}\hat\theta}\right|\left|\frac{\mathrm{d}\hat\theta}{\mathrm{d}t}\right|, \qquad \left|\frac{\mathrm{d}\hat\sigma(t)}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}\hat\sigma}{\mathrm{d}\hat\theta}\right|\left|\frac{\mathrm{d}\hat\theta}{\mathrm{d}t}\right|$$

$$\left|\frac{\mathrm{d}\hat\lambda(t)}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}\hat\lambda}{\mathrm{d}\hat\theta}\right|\left|\frac{\mathrm{d}\hat\theta}{\mathrm{d}t}\right|$$

where we have introduced the eccentric anomaly $\hat E(t)$ and the angle $\hat\theta(t) = \hat\nu(t) + \omega$, $\hat\nu(t)$ being the true anomaly. From classical astrodynamics, one can obtain

$$\left|\frac{\mathrm{d}\hat\theta}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}\hat\theta}{\mathrm{d}\hat E}\frac{\mathrm{d}\hat E}{\mathrm{d}t}\right| = b\sqrt{\frac{\mu}{a}}\frac{1}{\hat r^2}$$

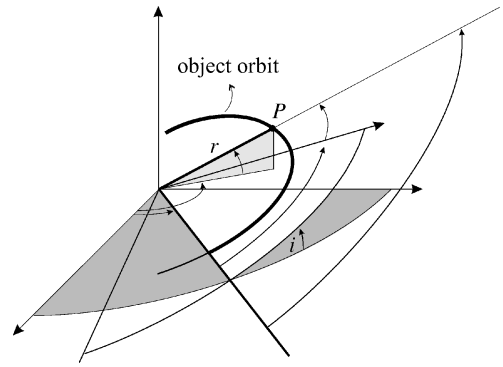

Fig. 1 Visualization of the orbital angles.

As an example, we evaluate $|\mathrm{d}\hat\lambda/\mathrm{d}\hat\theta|$. From the last part of Eq. (8), we get

$$\cos\hat\lambda\frac{\mathrm{d}\hat\lambda}{\mathrm{d}\hat\theta} = \sin i\cos\hat\theta$$

$$\cos\hat\theta = \pm\sqrt{1-(\sin\hat\lambda/\sin i)^2}$$

We therefore get

$$\left|\frac{\mathrm{d}\hat\lambda}{\mathrm{d}\hat\theta}\right| = \frac{\sqrt{\sin^2 i-\sin^2\hat\lambda}}{|\cos\hat\lambda|} = \frac{\sqrt{\sin^2 i-\sin^2\hat\lambda}}{\cos\hat\lambda}$$

A visualization of the geometrical meaning of all of the angles appearing in these equations is given in Fig. 1.

We then obtain

$$\left|\frac{\mathrm{d}\hat\lambda}{\mathrm{d}\hat\theta}\right| = \frac{\sqrt{\sin^2 i-\sin^2\hat\lambda}}{\cos\hat\lambda}, \qquad \left|\frac{\mathrm{d}\hat\sigma}{\mathrm{d}\hat\theta}\right| = \frac{|\cos i|}{\cos^2\hat\lambda}$$

$$\left|\frac{\mathrm{d}\hat r}{\mathrm{d}\hat\theta}\right| = \frac{\hat r}{p}\frac{\sqrt{(r_a-\hat r)(\hat r-r_p)}}{\sqrt{1-e^2}}$$

Having evaluated these derivatives, we are now able to eliminate a Dirac delta in Eq. (7) by randomizing with respect to time. The following three expressions are obtained depending on the eliminated Dirac function:

$$\rho(r,\sigma,\lambda|e,a,\omega,\Omega,i) = \frac{1}{2\pi}\frac{(1-e^2)^{\frac{3}{2}}}{\{1+e\cos[\hat\theta(\sigma)-\omega]\}^2}\frac{\cos^2\hat\lambda(\sigma)}{|\cos i|}\delta_r[r-\hat r(\sigma)]\delta_\lambda[\lambda-\hat\lambda(\sigma)]$$

$$\rho(r,\sigma,\lambda|e,a,\omega,\Omega,i) = \begin{cases}\dfrac{1}{2\pi a}\dfrac{r}{\sqrt{(r_a-r)(r-r_p)}}\displaystyle\sum_i\{\delta_\sigma[\sigma-\hat\sigma_i(r)]\delta_\lambda[\lambda-\hat\lambda_i(r)]\} & \text{if} \quad r_p<r<r_a \\ 0 & \text{otherwise}\end{cases}$$

$$\rho(r,\sigma,\lambda|e,a,\omega,\Omega,i) = \begin{cases}\dfrac{1}{2\pi ab}\dfrac{\cos\lambda}{\sqrt{\sin^2 i-\sin^2\lambda}}\displaystyle\sum_i\left\{\hat r_i^2(\lambda)\delta_\sigma[\sigma-\hat\sigma_i(\lambda)]\delta_r[r-\hat r_i(\lambda)]\right\} & \text{if} \quad \lambda^2<i^2 \\ 0 & \text{otherwise}\end{cases} \tag{9}$$

The remaining derivatives can be evaluated from the implicit definitions of the quantities $\hat r$, $\hat\sigma$, $\hat\lambda$ given by the following trigonometric identities:

$$\tan\hat\theta = \frac{\tan(\hat\sigma-\Omega)}{\cos i}, \qquad \tan\hat\lambda = \sin(\hat\sigma-\Omega)\tan i$$

$$\hat r = \frac{p}{1+e\cos(\hat\theta-\omega)}, \qquad \sin\hat\lambda = \sin i\sin\hat\theta \tag{8}$$

where $\hat\sigma_i(r)$ and $\hat\lambda_i(r)$ are the two possible longitudes and latitudes occupied by a satellite when its distance r is given and $\hat r_i(\lambda)$, $\hat\sigma_i(\lambda)$ are the two possible radii and longitudes occupied by a satellite when its latitude λ is given. The relation between the angle formed by the position vector and the line of nodes, and the longitude has been indicated with the symbol $\hat\theta(\sigma)$. The last two equations might look similar to Eqs. (7) and (36) in Dennis.[4] Formally those equations can be obtained by integrating in σ, λ the second of Eqs. (9) and by integrating in σ, r the third of Eqs. (9).

Starting as an example from the first expression in Eq. (9), we can choose two parameters to randomize, one out of the group a, e, ω, to eliminate the Dirac delta function δ_r and the other out of the group Ω, i to eliminate the Dirac delta function δ_λ. This leads to a total of six formulas that describe the statistical properties of the orbits. The number of possible expressions is really much larger as it also depends on the parameters we choose to describe the orbit.

Derivation of One of the New Expressions

The full derivation of each of the six formulas that can be obtained from Eq. (9) is not given here as it is simply an algebraic manipulation. Because of the definition of the argument of perigee ω and the longitude of the ascending node Ω [or the right ascension of the ascending node (RAAN) depending on the chosen inertial reference frame], these angles are probably the most suitable parameters to be randomized. The derivation of the PDF obtained by randomizing these parameters is fully outlined here, whereas the other formulas are just listed. We start from the first part of Eq. (9). In this equation the expressions $\hat{\theta}(\sigma)$, $\hat{r}(\sigma)$, $\hat{\lambda}(\sigma)$ are the functions relating a given longitude to the orbital position. Their implicit definition is given in Eq. (8).

The argument of perigee is then randomized. Its probability density function, defined in $[0, 2\pi]$, is indicated with f. To eliminate the Dirac delta $\delta_r[r - \hat{r}(\sigma)]$, we must evaluate the derivative

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}\omega} = \frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{\theta}}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\omega}$$

We get, from the third of Eq. (8), $\mathrm{d}\hat{r}/\mathrm{d}\omega = -\mathrm{d}\hat{r}/\mathrm{d}\hat{\theta}$; hence,

$$\rho_\omega(r, \sigma, \lambda | e, a, i, \Omega) = \begin{cases} \dfrac{1}{2\pi a}\dfrac{\cos^2\hat{\lambda}}{|\cos i|}\dfrac{r}{\sqrt{(r_a - r)(r - r_p)}}\displaystyle\sum_{i=1}^{2} f\left(\omega_i^*\right)\delta_\lambda[\lambda - \hat{\lambda}], & r_p < r < r_a \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

taking into account that the equation $r = \hat{r}$ has two solutions given by

$$\omega_i^* = (\{\pm\arccos[(p - r)/er] + \hat{\theta}(\sigma)\} \bmod (2\pi))$$

Note that this last expression depends on the longitude of the ascending node Ω and on the inclination i through the term $\hat{\theta}(\sigma)$ defined by the first part of Eq. (8). (The ambiguities in the quadrant can be resolved by observing that $\sigma - \Omega$ and $\hat{\theta}$ are in the same quadrant.) We now randomize Ω in Eq. (10) eliminating the Dirac delta δ_λ using Eq. (1). We therefore must evaluate an expression for $|\mathrm{d}\hat{\lambda}/\mathrm{d}\Omega|$ and for the solutions in Ω of the equation $\lambda = \hat{\lambda}$. It is possible, using Eq. (8), to show that

$$\left|\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}\Omega}\right| = \frac{\cos\hat{\lambda}}{|\cos i|\sqrt{\sin^2 i - \sin^2\hat{\lambda}}}$$

The solutions in Ω of the equation $\lambda = \hat{\lambda}$ can also be obtained from the relation $\tan\lambda = \sin(\sigma - \Omega)\tan i$:

$$\Omega_i^* = \{[\pm\arcsin(\tan\lambda/\tan i) + \sigma] \bmod (2\pi)\}$$

Table 1 Symbology adopted for the probability density functions

Probability density function	Relative parameter	Definition range
f	ω	$[0, 2\pi]$
g	Ω	$[0, 2\pi]$
G	e	$[0, 1]$
H	a	$[0, \infty]$
L	i	$[0, \pi]$

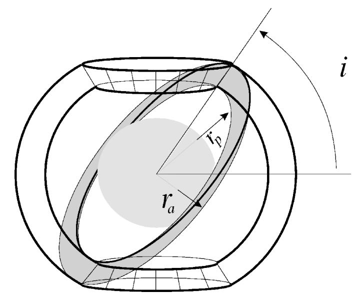

Fig. 2 Volume occupied by the satellite for values of ω and Ω.

This occurs when $\lambda^2 < i^2$. The final expression for the PDF can now be evaluated:

$$\rho_{\omega,\Omega}(r, \sigma, \lambda | e, a, i) = \begin{cases} \dfrac{1}{2\pi a}\dfrac{r\cos\lambda}{\sqrt{\sin^2 i - \sin^2\lambda}}\dfrac{\sum_i\sum_j g\left(\Omega_j^*\right) f\left[\omega_i^*\left(\Omega_j^*\right)\right]}{\sqrt{(r_a - r)(r - r_p)}} & \begin{cases} r_p < r < r_a \\ \lambda^2 < i^2 \end{cases} \\ 0 \end{cases} \tag{11}$$

A preliminary form of Eq. (11) appeared in the work by Valente and Izzo[2] and its application to the determination of the spatial density of LEO, GEO, and Molnyia satellites might be found in the recent work by Izzo.[1] When f and g are constant (equal to $1/2\pi$), the preceding expression can be compared to that obtained by combining Eqs. (7) and (36) appearing in Dennis.[4] A useful interpretation of this equation comes when considering the satellite as nonuniformly spread (or diffused to use Dennis' words) in a three-dimensional volume defined by $\lambda^2 < i$, $r_p < r < r_a$ (Fig. 2). The statistical properties of ω and Ω determine how the satellite mass is spread in the volume.

Six Final Expressions

Following the same mathematical derivation and changing the parameters to randomize, other important expressions can be obtained. Having chosen the spherical coordinates to describe the position of the orbiting body, and the osculating parameters to describe the orbit, a total of six possible formulas can be derived. These expressions are listed in this section. The generic PDF ρ will be conditioned by the parameters that follow the vertical bar (a classical symbolism in the theory of probabilities) and will consider the parameters they carry as subscripts to be random. The probability density functions describing the statistical properties of the various parameters are listed in Table 1. The list of the six formulas follows:

$$\rho_{e,i}(r, \sigma, \lambda | a, \omega, \Omega) = \begin{cases} \dfrac{1}{2\pi}\dfrac{L(i^*)G[e^*(i^*)]\cos\lambda}{\sqrt{1 - \cos^2(\sigma - \Omega)\cos^2\lambda}}\dfrac{[r^2\cos^2[\theta(\sigma) - \omega] - (p - r)^2]^{\frac{3}{2}}}{r^3\cos^4[\hat{\theta}(\sigma) - \omega]}, & 0 < e^* < 1 \\ 0 \end{cases}$$

$$\rho_{e,\Omega}(r, \sigma, \lambda | a, \omega, i) = \begin{cases} \dfrac{1}{2\pi}\dfrac{\sum_j g\left(\Omega_j^*\right)G\left[e^*\left(\Omega_j^*\right)\right]\cos\lambda}{\sqrt{\sin^2 i - \sin^2\lambda}}\dfrac{[r^2\cos^2[\theta(\sigma) - \omega] - (p - r)^2]^{\frac{3}{2}}}{r^3\cos^4[\hat{\theta}(\sigma) - \omega]}, & \begin{matrix} 0 < e^* < 1 \\ \lambda^2 < i^2 \end{matrix} \\ 0 \end{cases}$$

$$\rho_{a,i}(r, \sigma, \lambda | e, \omega, \Omega) = \frac{1}{2\pi}\frac{L(i^*)H[a^*(i^*)]\cos\lambda}{\sqrt{1 - \cos^2(\sigma - \Omega)\cos^2\lambda}}\frac{\sqrt{(1 - e^2)^5}}{1 + e\cos[\hat{\theta}(\sigma) - \omega)]}$$

$$\rho_{a,\Omega}(r,\sigma,\lambda|e,\omega,i)=\begin{cases}\dfrac{1}{2\pi}\dfrac{\sum_j g\left(\Omega_j^*\right)H\left[a^*\left(\Omega_j^*\right)\right]\cos\lambda}{\sqrt{\sin^2 i-\sin^2\lambda}}\dfrac{\sqrt{(1-e^2)^5}}{1+e\cos[\hat\theta(\sigma)-\omega]}, & \lambda^2<i^2\\ 0\end{cases}$$

$$\rho_{\omega,\Omega}(r,\sigma,\lambda|e,a,i)=\begin{cases}\dfrac{1}{2\pi a}\dfrac{r\cos\lambda}{\sqrt{\sin^2 i-\sin^2\lambda}}\dfrac{\sum_i\sum_j g\left(\Omega_j^*\right)f\left[\omega_i^*\left(\Omega_j^*\right)\right]}{\sqrt{(r_a-r)(r-r_p)}}, & \begin{cases}r_p<r<r_a\\ \lambda^2<i^2\end{cases}\\ 0\end{cases}$$

$$\rho_{\omega,i}(r,\sigma,\lambda|e,a,\Omega)=\begin{cases}\dfrac{1}{2\pi a}\dfrac{r\cos\lambda}{\sqrt{1-\cos^2(\sigma-\Omega)\cos^2\lambda}}\dfrac{\sum_j f\left[\omega_j^*(i^*)\right]L(i^*)}{\sqrt{(r_a-r)(r-r_p)}}, & r_p<r<r_a\\ 0\end{cases}$$

To evaluate these formulas in any point of the three-dimensional space, we must first find the various e^*, Ω^*, etc. It is convenient to start evaluating Ω^* or i^*. This can be easily done with the aid of Table 2.

It is then possible to proceed with the evaluation of e^*, a^*, or ω^* that will depend upon Ω^* or i^* through the expression $\hat\theta(\sigma)$. Table 3 shows all of the expressions that are used.

From the Probability Density Functions to Probability

The equalities that have been established provide solutions related to the probabilistic behavior of satellites. Some examples show how some classical results can be easily derived and extended by performing some quadratures.

Consider a single satellite in a known orbit. What is the probability of this satellite being at a certain distance range from the center of the Earth? To not speak about a deterministic process, the time instant is considered to be randomized, or equivalently the time of pericenter passage. The answer is easily given by the following equalities:

$$P(r_1<r<r_2)=\int_{r_1}^{r_2}\rho(r|a,e,\omega,\Omega,i)\,\mathrm{d}r$$

where, from the second of Eq. (9),

$$\rho(r|e,a,\omega,\Omega,i)=\int_0^{2\pi}\int_0^{2\pi}\rho(r,\sigma,\lambda|a,e,\omega,\Omega,i)\,\mathrm{d}\lambda\,\mathrm{d}\sigma$$

$$=\begin{cases}\dfrac{1}{\pi a}\dfrac{r}{\sqrt{(r_a-r)(r-r_p)}} & \text{if}\quad r_p<r<r_a\\ 0 & \text{otherwise}\end{cases}$$

Table 2 Expressions for Ω^*, i^*

Parameter	Equation	Number of solutions
i^*	$\tan i^*=\dfrac{\tan\lambda}{\sin(\sigma-\Omega)}$	1
Ω^*	$\sin(\sigma-\Omega^*)=\dfrac{\tan\lambda}{\tan i}$	2(0)

Table 3 Expressions for a^*, e^*, ω^*

Parameter	Equation	Number of solutions
a^*	$a^*=\dfrac{r\{1+e\cos[\hat\theta(\sigma)-\omega]\}}{1-e^2}$	1
e^*	$e^*=\dfrac{p-r}{r\cos[\hat\theta(\sigma)-\omega]}$	1(0)
ω^*	$\cos[\omega^*-\hat\theta(\sigma)]=\dfrac{p-r}{er}$	2(0)

that, after integration and some algebraic manipulations, leads to the well-known expression [see Chobotov,[5] Eq. (13.5), p. 348]

$$P(r_1<r<r_2)=(1/\pi)\arcsin[2(r-a)/(r_a-r_p)]$$
$$-(1/\pi a)\sqrt{(r_a-r)(r-r_p)}\Big|_{r_{\min}}^{r_{\max}}$$

Let us now find the probability of a given satellite being at a certain longitude range. We start by writing the expression

$$P(\sigma_1<\sigma<\sigma_2)=\int_{\sigma_1}^{\sigma_2}\rho(\sigma|a,e,\omega,\Omega,i)\,\mathrm{d}\sigma$$

where, from the first part of Eq. (9),

$$P(\sigma_1<\sigma<\sigma_2)=\int_{\sigma_1}^{\sigma_2}\frac{1}{2\pi}\frac{\sqrt{(1-e^2)^{\frac{3}{2}}}}{\{1+e\cos[\hat\theta(\sigma)-\omega]\}^2}\frac{\cos^2\hat\lambda(\sigma)}{|\cos i|}\,\mathrm{d}\sigma$$

Now consider finding the probability of a given satellite being at a certain latitude range. As before, we start from the expression

$$P(\lambda_1<\lambda<\lambda_2)=\int_{\lambda_1}^{\lambda_2}\rho(\lambda|a,e,\omega,\Omega,i)\,\mathrm{d}\lambda$$

and, using the third part of Eq. (9) we obtain

$$P(\lambda_1<\lambda<\lambda_2)$$
$$=\begin{cases}\displaystyle\int_{\sigma_1}^{\sigma_2}\frac{1}{2\pi ab}\frac{[\hat r_1^2(\lambda)+\hat r_2^2(\lambda)]\cos\lambda}{\sqrt{\sin^2 i-\sin^2\lambda}}\,\mathrm{d}\sigma & \text{if}\quad \lambda^2<i^2\\ 0 & \text{otherwise}\end{cases}$$

where r_1, r_2 are the two possible radii occupied by the satellite when its latitude is given. [This expression is similar to Eq. (36) in Dennis.[4] The methodology used here allows us to consider orbits arbitrarily positioned in the three-dimensional space. In the particular cases $\omega=\pi/2$ or $3/4\pi$, this expression coincides with Dennis'.] In the simple case of a zero eccentricity orbit, the preceding integral can be evaluated by the substitution $t=\sin\lambda$.

In the preceding examples the argument of perigee and the longitude of the ascending node were considered to be fixed. This is true for a very short period of time, maybe a few days, but after that the Earth oblateness and other perturbations will have changed these parameters considerably. Note that the other parameters change, but in a much slower way. This justifies, as it has been already mentioned, a more detailed analysis of the case with ω and Ω randomized. In this case the probabilities of finding the satellite in a certain radius, longitude, or latitude range must be evaluated again. The following will, by definition, answer these problems:

$$P(r_1 < r < r_2) = \int_{r_1}^{r_2} \rho_{\omega,\Omega}(r|e,a,i)\,\mathrm{d}r$$

$$P(\sigma_1 < \sigma < \sigma_2) = \int_{\sigma_1}^{\sigma_2} \rho_{\omega,\Omega}(\sigma|e,a,i)\,\mathrm{d}\sigma$$

$$P(\lambda_1 < \lambda < \lambda_2) = \int_{\lambda_1}^{\lambda_2} \rho_{\omega,\Omega}(\lambda|e,a,i)\,\mathrm{d}\lambda$$

where the various ρ are obtained by integrating Eq. (11). We finally obtain a triple integral that must be evaluated numerically when the functions f and g are generic. If $g = 1/2\pi$ and f is a generic normalized function, the integral in λ can be evaluated analytically, and only two numerical quadratures are needed. (The same happens if $f = 1/2\pi$ and g is a generic normalized function.) In the case in which the two functions f, g can be both considered identically equal to $1/2\pi$, all of the integrals can by solved analytically, leading again to known results.

Application: Object Spatial Density for Molniya Satellites

The Molniya satellite environment is now considered for the new expressions. Using Eq. (11) to model the spatial density of a single satellite, we can define the object spatial density[1] as

$$\rho_{\mathrm{osd}}(r,\sigma,\lambda) = \sum_{j=1}^{n} \rho_{\omega,\Omega}(r,\sigma,\lambda|\hat{e}_j,\hat{a}_j,\hat{i}_j) \tag{12}$$

This is no longer a PDF (its integral over the entire space is not one); it is just an auxiliary function useful in the evaluation of probabilities related to the totality of the objects belonging to a given family. We observe that the majority of the Molniya satellites have their perigee at the highest possible latitude, allowing these satellites to appear to hover for most of their lifetime at a high latitude (63°). The orbital parameter ω will, as a consequence, have a high probability to assume a value around 270 deg. The statistical description of this environment cannot be based on an assumption of uniform probability density functions. The approach proposed here is therefore very appropriate for this case. In Fig. 3 the probability density functions f and g are shown for the Molniya environment. The USSPACECOM catalog relative to epoch 05/09/2002 was used to extract these functions, and the probability distribution functions were built considering intervals of 5 deg. Note that f, as we were expecting, is not constant. On the other hand, Ω can be considered to be much more uniformly distributed and therefore g resembles a constant function.

The evaluation of the object spatial density in one point of the space through Eq. (12) requires, for the Molnyia satellites, 303 computations of Eq. (11). The results are shown in Figs. 4 and 5, where also the results obtained by using Dennis' approach are shown. In particular the graph in Fig. 4 shows the object spatial density plotted against the latitude for various longitudes and radii. We observe here that, being the Molnyia orbits characterized by a simple distribution of the RAAN, the longitude σ affects the object spatial density, although very little.

The latitude is, on the other hand, a crucial variable, so that at 40,000 km there is little chance to find a Molniya satellite in the southern hemisphere. The following graph shows the object spatial density plotted against the radius. Because the speed of a satellite is the lowest possible at the apogee, this region contains the highest probability of finding a Molniya satellite. The graph shows a uniform trend as the growth of the integration element $\mathrm{d}\sigma\,\mathrm{d}\lambda\,\mathrm{d}r$ has to be accounted for. The results obtained by applying the Dennis equations to the same problem show how the hypotheses of his model lead to symmetry between the object spatial density in the northern and in southern hemisphere. In fact, considering the argument of perigee equally distributed in the interval $[0, 2\pi]$ makes the object spatial density be a symmetric function of λ. On the other hand, the approach based on Eqs. (11) and (12) accounts for the differences between northern and southern latitudes. Hence the discrepancies in Fig. 4 between the two approaches.

Interesting results might also be found investigating the object spatial density for LEO and GEO satellites. In the work by Izzo,[1] Eqs. (11) and (12) are shown to be able to account for the clustering of GEO satellites in some portions of the geostationary ring and to reveal a greater density of LEO satellites in the northern hemisphere. It can be argued that the assumption of Keplerian motion is quite restrictive and that the object spatial density derived might be affected by this hypothesis. Perturbations do, of course, modify the motion of a single satellite, but the effects of perturbations on the overall object spatial density decrease with the number of satellites considered. As a result, the PDF describing the statistical properties of a given family of orbits tend to be quite stable over time.[3] Future work on this subject might try to include the effect of perturbations on the statistical description of a single satellite, in which case the issue of determining the harmonic period for satellite motion would be a key feature.

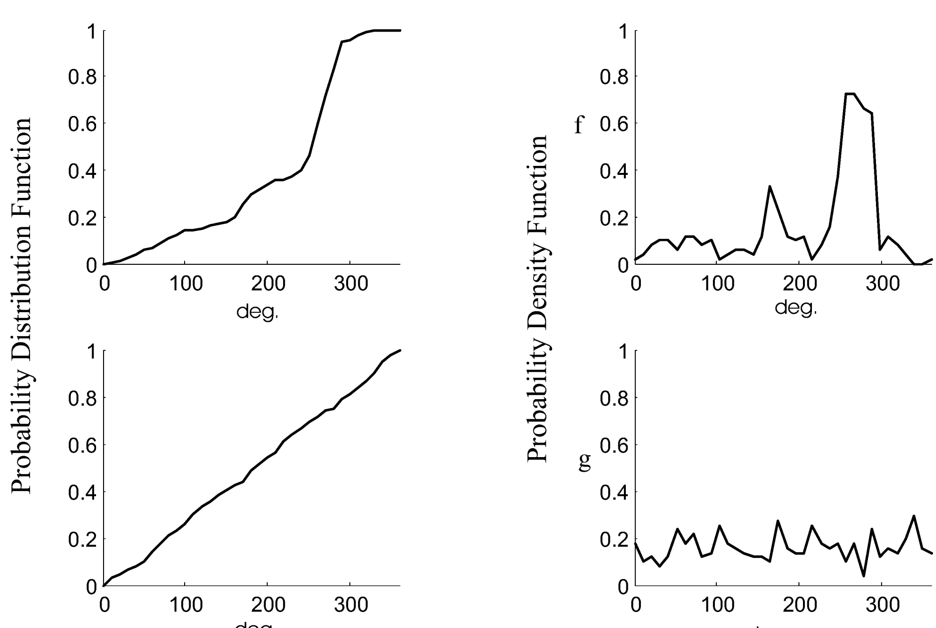

Fig. 3 Statistical properties of the Molnyia environment.

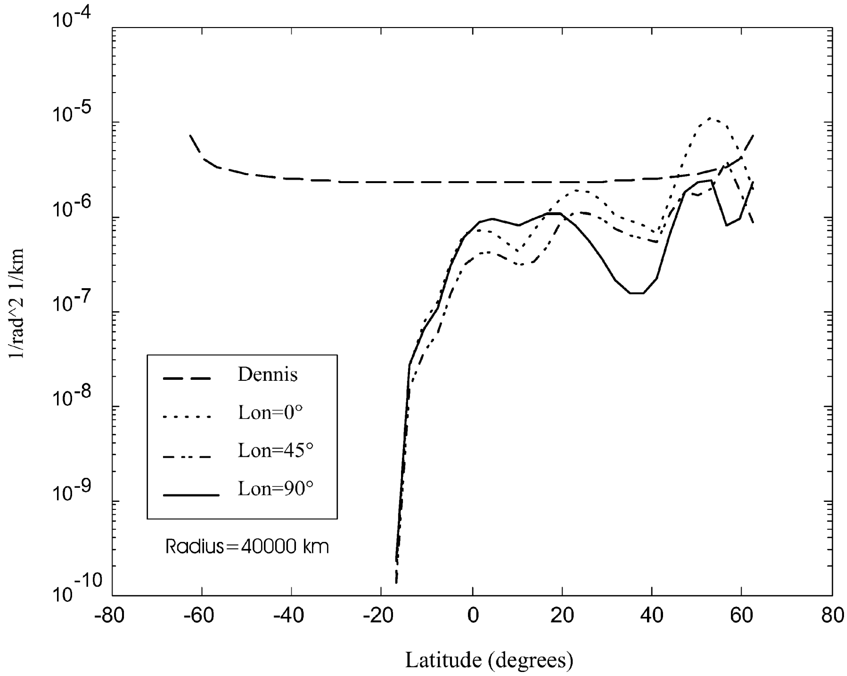

Fig. 4 Object spatial density for Molnyia environment vs latitude.

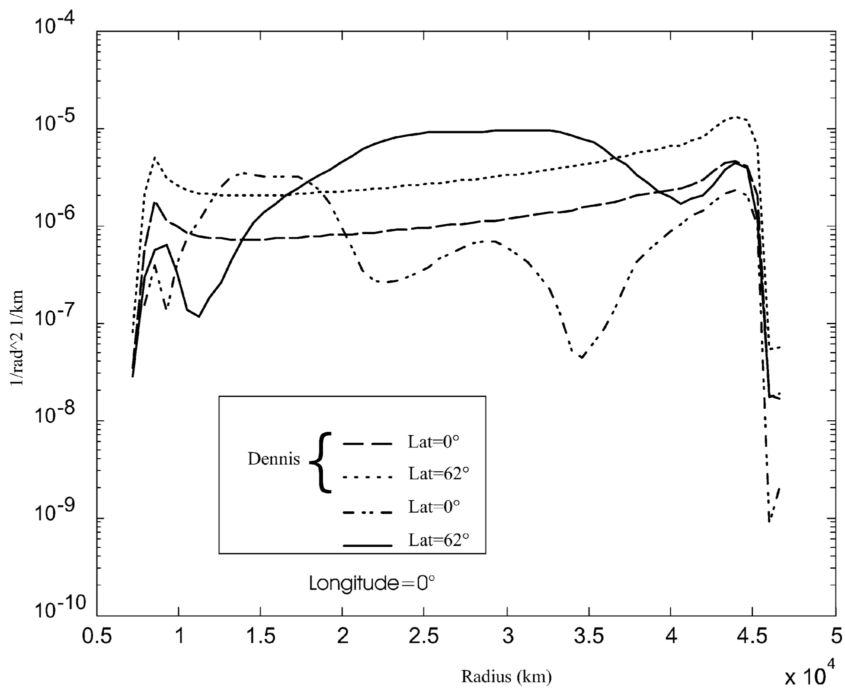

Fig. 5 Object spatial density for Molnyia environment vs radius.

Conclusions

A new set of equations describing the statistical properties of a Keplerian orbit has been derived. The new equations describe situations of uncertainties on the classical Keplerian set of parameters. The theory that results is shown to be a generalization of Dennis equations. This generalization allows one to account for nonuniform distributions of the orbital parameters, and it might be particularly useful in the description of the object spatial density for satellites in Molnyia and geostationary orbits. The derivation of the new expressions makes use of a novel methodology based on some properties of the Dirac delta, which simplifies the standard change of variable method used in probabilities.

References

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